

ASYMPTOTIC BEHAVIOR OF THE LEAST COMMON MULTIPLE OF CONSECUTIVE ARITHMETIC PROGRESSION TERMS

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ABSTRACT. Let l and m be two integers with $l > m \geq 0$, and let a and b be integers with $a \geq 1$ and $a + b \geq 1$. In this paper, we prove that $\log \text{lcm}_{mn < i \leq ln} \{ai + b\} = An + o(n)$, where A is a constant depending on l, m and a .

1. INTRODUCTION

Chebyshev [2] initiated the study of the least common multiple of consecutive positive integers for the first significant attempt to prove prime number theorem. Actually, Chebyshev introduced the function $\Psi(x) = \sum_{p^k \leq x} \log p = \log \text{lcm}_{1 \leq i \leq x} \{i\}$, and then obtained an equivalent form of the prime number theorem, which states that $\log \text{lcm}(1, \dots, n) \sim n$ as n goes to infinity. Since then, this topic received many authors' attention. Hanson [5] and Nair [9] got upper and lower bounds of $\text{lcm}_{1 \leq i \leq n} \{i\}$, respectively. Farhi [3] investigated the least common multiple of arithmetic progressions, while Farhi and Kane [4] and Hong and Yang [7] studied the least common multiple of consecutive positive integers. Hong and Qian [6] obtained some results on the least common multiple of consecutive arithmetic progression terms. Let a and b be integers such that $a \geq 1$, $a + b \geq 1$ and $\gcd(a, b) = 1$. On the other hand, Bateman, Kalb and Stenger [1] proved that

$$\log \text{lcm}_{1 \leq i \leq n} \{ai + b\} \sim \frac{an}{\varphi(a)} \sum_{\substack{r=1 \\ \gcd(r, a)=1}}^a \frac{1}{r}$$

as $n \rightarrow \infty$, where $\varphi(a)$ denotes the number of integers relatively prime to a between 1 and a . Farhi [3] showed that $\text{lcm}_{1 \leq i \leq n} \{i^2 + 1\} \geq 0.32 \cdot (1.442)^n$ for all $n \geq 1$. Qian, Tan and Hong [10] proved that for any given positive integer k , $\log \text{lcm}_{0 \leq i \leq k} \{(n+i)^2 + 1\} \sim 2(k+1) \log n$ as $n \rightarrow \infty$.

In this paper, we mainly focus on the least common multiple

$$\text{lcm}_{mn < i \leq ln} \{ai + b\}$$

of consecutive arithmetic progression terms, where l and m are integers such that $l > m \geq 0$ and $a \geq 1$ and b are integers such that $a + b \geq 1$. Evidently, if $\gcd(a, b) = d$, then

$$\begin{aligned} \log \text{lcm}_{mn < i \leq ln} \{ai + b\} &= \log \text{lcm}_{mn < i \leq ln} \{a_1 i + b_1\} + \log d \\ &= \log \text{lcm}_{mn < i \leq ln} \{a_1 i + b_1\} + O(1), \end{aligned}$$

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where $a_1 = a/d$ and $b_1 = b/d$. So it is sufficient to consider the case $\gcd(a, b) = 1$. Let h and k be any given two relatively prime positive integers. We define

$$(1.1) \quad \vartheta(x; h, k) := \sum_{\substack{\text{prime } p \leq x \\ p \equiv k \pmod{h}}} \log p.$$

Then the prime number theorem for arithmetic progressions states that

$$(1.2) \quad \vartheta(x; h, k) = \frac{x}{\varphi(h)} + O(x \exp(-c_1 \sqrt{\log x})),$$

where $c_1 > 0$ is a constant (see, for example, [8]).

As usual, for any prime p , we let v_p be the normalized p -adic valuation of \mathbb{N}^* , i.e., $v_p(a) = s$ if $p^s \parallel a$. For any real number x , we let $\lfloor x \rfloor$ denote the largest integer no more than x . We can now state the main result of this paper.

Theorem 1.1. *Let l and m be integers with $l > m \geq 0$ and let $a \geq 1$ and b be integers such that $a + b \geq 1$ and $\gcd(a, b) = 1$. Then*

$$\log \text{lcm}_{mn < i \leq ln} \{ai + b\} \sim \frac{an}{\varphi(a)} \sum_{\substack{r=1 \\ \gcd(r, a)=1}}^a A_r$$

as $n \rightarrow \infty$, where

$$(1.3) \quad A_r := \begin{cases} \frac{l}{r}, & \text{if } l \geq \frac{(a+r)m}{r}, \\ \sum_{i=0}^{K_r-1} \frac{l-m}{r+ai} + \frac{l}{r+aK_r}, & \text{if } m < l < \frac{(a+r)m}{r} \end{cases}$$

with

$$(1.4) \quad K_r := \left\lfloor \frac{al - (l-m)r}{a(l-m)} \right\rfloor.$$

From Theorem 1.1, we can deduce immediately the following interesting results.

Corollary 1.2. *Let l and m be integers with $l > m \geq 0$ and let a and b be integers such that $a \geq 1$, $a + b \geq 1$ and $\gcd(a, b) = 1$. If $l \geq (a+1)m$, then*

$$\log \text{lcm}_{mn < i \leq ln} \{ai + b\} \sim \frac{aln}{\varphi(a)} \sum_{\substack{r=1 \\ \gcd(r, a)=1}}^a \frac{1}{r}$$

as $n \rightarrow \infty$.

Taking $l = 1$ and $m = 0$, Corollary 1.2 then becomes the Bateman-Kalb-Stenger theorem [1].

Corollary 1.3. *Let l and m be integers with $l > m \geq 0$. Then*

$$\log \text{lcm}_{mn < i \leq ln} \{i\} \sim Bn$$

as $n \rightarrow \infty$, where

$$B = \begin{cases} l, & \text{if } l \geq 2m, \\ \frac{l}{\lfloor \frac{l}{l-m} \rfloor} + (l-m) \sum_{i=1}^{\lfloor \frac{l-m}{l-m} \rfloor} \frac{1}{i}, & \text{if } m < l < 2m. \end{cases}$$

The next section will devote to the proof of Theorem 1.1.

2. PROOF OF THEOREM 1.1

In this section, we show Theorem 1.1.

Proof of Theorem 1.1. For simplicity, we let

$$L_{m,l}(n) := \text{lcm}_{mn < i \leq ln} \{b + ai\}$$

and by $P_{m,l}(n)$ we denote the set of all the prime factors of $L_{m,l}(n)$. Denote by $R(a)$ the set of all the integers relatively prime to a between 1 and a .

For any given integer b with $a+b \geq 1$ and $\gcd(a, b) = 1$, we can assume that $b = b_0 + qa$ for some integers q and $b_0 \in R(a)$. Then

$$\begin{aligned} \log \text{lcm}(b + a(mn + 1), \dots, b + aln) &= \log \text{lcm}(b_0 + a(mn + 1 + q), \dots, b_0 + a(ln + q)) \\ &= \log \text{lcm}_{mn < i \leq ln} \{b_0 + ai\} + O(\log n) \\ &= \log \text{lcm}_{mn < i \leq ln} \{b_0 + ai\} + o(n). \end{aligned}$$

So to prove Theorem 1.1, we may assume that $b \in R(a)$ in what follows.

First of all, we have

$$L_{m,l}(n) = \prod_{p \in P_{m,l}(n)} p^{v_p(L_{m,l}(n))} = \left(\prod_{p \in P_{m,l}(n)} p \right) \left(\prod_{\substack{p \in P_{m,l}(n) \\ p^2 | L_{m,l}(n)}} p^{v_p(L_{m,l}(n)) - 1} \right).$$

So

$$(2.1) \quad \log L_{m,l}(n) = \sum_{p \in P_{m,l}(n)} \log p + \sum_{\substack{p \in P_{m,l}(n) \\ p^2 | L_{m,l}(n)}} (v_p(L_{m,l}(n)) - 1) \log p.$$

If $p^2 | L_{m,l}(n)$, then $p^2 | (b + ai_0)$ for some integer $mn < i_0 \leq ln$, which implies that $p^2 \leq b + ai_0 \leq b + aln$. Hence for any prime p running over the second summation in (2.1), we have $p \leq \sqrt{b + aln}$. Since $p^{v_p(L_{m,l}(n))} \leq b + aln$ for any $p \in P_{m,l}(n)$, one has

$$v_p(L_{m,l}(n)) \leq \frac{\log(b + aln)}{\log p}.$$

It follows from the prime number theorem that

$$\begin{aligned} \sum_{\substack{p \in P_{m,l}(n) \\ p^2 | L_{m,l}(n)}} (v_p(L_{m,l}(n)) - 1) \log p &\leq \sum_{p \leq \sqrt{b + aln}} \frac{\log(b + aln)}{\log p} \log p = \sum_{p \leq \sqrt{b + aln}} \log(b + aln) \\ &\ll \frac{\sqrt{b + aln}}{\log \sqrt{b + aln}} \log(b + aln) = 2\sqrt{b + aln}. \end{aligned}$$

Then by (2.1) we obtain

$$(2.2) \quad \log L_{m,l}(n) = \sum_{p \in P_{m,l}(n)} \log p + O(\sqrt{n}).$$

To estimate $\sum_{p \in P_{m,l}(n)} \log p$, we first need to find a characterization on the primes in the set $P_{m,l}(n)$. Since $\gcd(a, b) = 1$, we get that $\gcd(a, b + ai) = 1$ for all positive integers

i , which implies that each prime p in $P_{m,l}(n)$ is relatively prime to a . Then each prime $p \in P_{m,l}(n)$ is congruent to some $r \in R(a)$ modulo a . It then follows from (2.2) that

$$(2.3) \quad \log L_{m,l}(n) = \sum_{r \in R(a)} \sum_{\substack{p \in P_{m,l}(n) \\ p \equiv r \pmod{a}}} \log p + O(\sqrt{n}).$$

By (2.3), to estimate $\log L_{m,l}(n)$, it suffices to estimate the sum

$$(2.4) \quad \sum_{\substack{p \in P_{m,l}(n) \\ p \equiv r \pmod{a}}} \log p$$

for each $r \in R(a)$, which will be done in the following.

Now fix an $r \in R(a)$. Then there exists only one $r' \in R(a)$ such that $rr' \equiv b \pmod{a}$. If p is a prime with $p \equiv r \pmod{a}$, then $r'p \equiv b \pmod{a}$. So any term divisible by p in the arithmetic progression $\{b + ai\}_{i=1}^{\infty}$ must be of the form $(r' + aj)p$ with $j \geq 0$ being an integer. Thus for any prime $p \equiv r \pmod{a}$, we have that $p \in P_{m,l}(n)$ if and only if there exists a nonnegative integer i_0 such that

$$b + amn < (r' + ai_0)p \leq b + aln.$$

In other words, a prime p congruent to r modulo a is in $P_{m,l}(n)$ if and only if

$$\frac{b + amn}{r' + ai_0} < p \leq \frac{b + aln}{r' + ai_0}$$

for some nonnegative integer i_0 . Therefore

$$(2.5) \quad \left\{ p \in P_{m,l}(n) : p \equiv r \pmod{a} \right\} = \bigcup_{i=0}^{\infty} \left\{ \text{prime } q \equiv r \pmod{a} : \frac{b + amn}{r' + ai} < q \leq \frac{b + aln}{r' + ai} \right\}.$$

In what follows we transfer the union in the right-hand side of (2.5) into a union of finitely many sets.

If $p \nmid a$, then exactly one term of any consecutive p terms in the arithmetic progression $\{b + ai\}_{i=1}^{\infty}$ is divisible by p . Therefore, for any prime p with $p \leq (l - m)n$ and $p \nmid a$, there is at least one integer i_0 such that $mn < i_0 \leq ln$ and $b + ai_0 \equiv 0 \pmod{p}$. Hence the primes p satisfying $p \leq (l - m)n$ and $p \nmid a$ are all in the set $P_{m,l}(n)$.

For convenience, we let $H = K_{r'}$, where $K_{r'}$ is defined as in (1.4). Then we have

$$(2.6) \quad H = \left\lfloor \frac{al - (l - m)r'}{a(l - m)} \right\rfloor.$$

Thus for any positive integer n , one has

$$(2.7) \quad (l - m)n < \frac{b + aln}{r' + aH}.$$

and for sufficiently large n , we have

$$\frac{b + aln}{r' + a(H + 1)} < (l - m)n.$$

In the rest of the proof, we always assume that n is a sufficient large integer.

Since $\frac{b + aln}{r' + ai}$ strictly decreases as i increases, it follows immediately that for any integer $i > H$, we have

$$(l - m)n > \frac{b + aln}{r' + ai} > \frac{b + amn}{r' + ai},$$

which implies that

$$\left(\frac{b+amn}{r'+ai}, \frac{b+aln}{r'+ai}\right] \subseteq (0, (l-m)n]$$

for any integer $i > H$. Therefore

$$\begin{aligned} & \bigcup_{i=H+1}^{\infty} \left\{ \text{prime } q \equiv r \pmod{a} : \frac{b+amn}{r'+ai} < q \leq \frac{b+aln}{r'+ai} \right\} \\ & \subseteq \left\{ \text{prime } q \equiv r \pmod{a} : q \leq (l-m)n \right\}. \end{aligned}$$

Hence by (2.5),

(2.8)

$$\begin{aligned} \{p \in P_{m,l}(n) : p \equiv r \pmod{a}\} &= \bigcup_{i=0}^H \left\{ \text{prime } q \equiv r \pmod{a} : \frac{b+amn}{r'+ai} < q \leq \frac{b+aln}{r'+ai} \right\} \\ &\quad \bigcup \left\{ \text{prime } q \equiv r \pmod{a} : q \leq (l-m)n \right\}. \end{aligned}$$

To estimate the sum (2.4), we consider the following two cases.

Case 1. $l > \frac{(a+r')m}{r'}$. Then $(a+r')m < lr'$. It follows from (2.6) that $H = 0$. So by (2.8), we only need to deal with the following union

(2.9)

$$\left\{ \text{prime } q \equiv r \pmod{a} : \frac{b+amn}{r'} < q \leq \frac{b+aln}{r'} \right\} \bigcup \left\{ \text{prime } q \equiv r \pmod{a} : q \leq (l-m)n \right\},$$

which is equal to the set $\{p \in P_{m,l}(n) : p \equiv r \pmod{a}\}$.

Since $(a+r')m < lr'$, we have

$$b+amn < (l-m)r'n + b \leq b+aln,$$

which implies that

$$\frac{b+amn}{r'} < (l-m)n + \frac{b}{r'} \leq \frac{b+aln}{r'}.$$

Then the union of the following three intervals

$$(0, (l-m)n] \bigcup ((l-m)n, (l-m)n + \frac{b}{r'}] \bigcup \left(\frac{b+amn}{r'}, \frac{b+aln}{r'}\right]$$

equals the interval $(0, \frac{b+aln}{r'}]$. Hence by (2.9), we obtain that all the primes $p \leq \frac{b+aln}{r'}$ congruent to r modulo a and not belonging to the interval $((l-m)n, (l-m)n + \frac{b}{r'}]$ are contained in the set $\{p \in P_{m,l}(n) : p \equiv r \pmod{a}\}$. So from (1.1) and (1.2), we derive that if $l > \frac{(a+r')m}{r'}$, then

$$\begin{aligned} (2.10) \quad \sum_{\substack{p \in P_{m,l}(n) \\ p \equiv r \pmod{a}}} \log p &= \sum_{\substack{\text{prime } p \equiv r \pmod{a} \\ p \leq \frac{b+aln}{r'}}} \log p + O\left(\sum_{\substack{\text{prime } p \equiv r \pmod{a} \\ (l-m)n < p \leq (l-m)n + \frac{b}{r'}}} \log p\right) \\ &= \vartheta\left(\frac{b+aln}{r'}; a, r\right) + O(\log n) \\ &= \frac{a}{\varphi(a)} \frac{ln}{r'} + O\left(n \exp(-c_1 \sqrt{\log n})\right). \end{aligned}$$

Case 2. $l \leq \frac{(a+r')m}{r'}$. In this case, $H \geq 1$. Clearly, $H > \frac{al-(l-m)r'}{a(l-m)} - 1$ by (2.6). So we obtain

$$\begin{aligned} (l-m)n - \frac{b+amn}{r'+aH} &= \frac{(l-m)r'n + aH(l-m)n - (b+amn)}{r'+aH} \\ &> \frac{(l-m)r'n + a(l-m)(\frac{al-(l-m)r'}{a(l-m)} - 1)n - (b+amn)}{r'+aH} \\ &= \frac{-b}{r'+aH}. \end{aligned}$$

It follows from $b \in R(a)$ and $r' \in R(a)$ that

$$(2.11) \quad \frac{b+amn}{r'+aH} < (l-m)n + \frac{b}{r'+aH} < (l-m)n + 1.$$

On the other hand, it is obvious that

$$\left\{ \text{prime } q \equiv r \pmod{a} : q \leq (l-m)n \right\} = \left\{ \text{prime } q \equiv r \pmod{a} : q < (l-m)n + 1 \right\}.$$

Thus we derive from (2.7) and (2.11) that

$$\begin{aligned} &\left\{ \text{prime } q \equiv r \pmod{a} : q \leq (l-m)n \right\} \cup \left\{ \text{prime } q \equiv r \pmod{a} : \frac{b+amn}{r'+aH} < q \leq \frac{b+aln}{r'+aH} \right\} \\ &= \left\{ \text{prime } q \equiv r \pmod{a} : q \leq \frac{b+aln}{r'+aH} \right\}. \end{aligned}$$

Then by (2.8), we have

$$(2.12) \quad \left\{ \text{prime } p \in P_{m,l}(n) : p \equiv r \pmod{a} \right\} = \mathcal{A}_1 \cup \mathcal{A}_2,$$

where

$$\mathcal{A}_1 = \bigcup_{i=0}^{H-1} \left\{ \text{prime } q \equiv r \pmod{a} : \frac{b+amn}{r'+ai} < q \leq \frac{b+aln}{r'+ai} \right\}$$

and

$$\mathcal{A}_2 = \left\{ \text{prime } q \equiv r \pmod{a} : q \leq \frac{b+aln}{r'+aH} \right\}.$$

Since $H = \lfloor \frac{al-(l-m)r'}{a(l-m)} \rfloor$, we have $al - (l-m)r' \geq aH(l-m)$. Thus for any positive integer $i \leq H$, we obtain

$$\begin{aligned} \frac{b+amn}{r'+a(i-1)} - \frac{b+aln}{r'+ai} &= \frac{a(b+aln - (l-m)r'n - ai(l-m)n)}{(r'+a(i-1))(r'+ai)} \\ &> \frac{a(aH(l-m)n - ai(l-m)n)}{(r'+a(i-1))(r'+ai)} \\ &= \frac{a^2(H-i)(l-m)n}{(r'+a(i-1))(r'+ai)} \geq 0. \end{aligned}$$

That is, for any positive integer $i \leq H$, one has

$$\frac{b+amn}{r'+a(i-1)} > \frac{b+aln}{r'+ai}.$$

It follows that for any integer $1 \leq i \leq H$, the intersection of

$$\left\{ \text{prime } q \equiv r \pmod{a} : \frac{b+amn}{r'+ai} < q \leq \frac{b+aln}{r'+ai} \right\}$$

and

$$\left\{ \text{prime } q \equiv r \pmod{a} : \frac{b + amn}{r' + a(i-1)} < q \leq \frac{b + aln}{r' + a(i-1)} \right\}$$

is empty and $\mathcal{A}_1 \cap \mathcal{A}_2$ is empty too.

First using (2.12), and then using (1.1) and (1.2), we obtain that if $l \leq \frac{(a+r')m}{r'}$, then

$$\begin{aligned} (2.13) \quad & \sum_{\substack{p \in P_{m,l}(n) \\ p \equiv r \pmod{a}}} \log p \\ &= \sum_{p \in \mathcal{A}_1} \log p + \sum_{p \in \mathcal{A}_2} \log p = \sum_{i=0}^{H-1} \sum_{\substack{p \equiv r \pmod{a} \\ \frac{b+amn}{r'+ai} < p \leq \frac{b+aln}{r'+ai}}} \log p + \sum_{\substack{p \equiv r \pmod{a} \\ p \leq \frac{b+aln}{r'+aH}}} \log p \\ &= \sum_{i=0}^{H-1} \left(\vartheta\left(\frac{b+aln}{r'+ai}; a, r\right) - \vartheta\left(\frac{b+amn}{r'+ai}; a, r\right) \right) + \vartheta\left(\frac{b+aln}{r'+aH}; a, r\right) \\ &= \sum_{i=0}^{H-1} \frac{1}{\varphi(a)} \frac{a(l-m)n}{r'+ai} + \frac{1}{\varphi(a)} \frac{aln}{r'+aH} + O(n \exp(-c_1 \sqrt{\log n})) \\ &= \frac{an}{\varphi(a)} \left(\sum_{i=0}^{H-1} \frac{l-m}{r'+ai} + \frac{l}{r'+aH} \right) + O(n \exp(-c_1 \sqrt{\log n})). \end{aligned}$$

Evidently, one has $H = 1$ and $\frac{l}{r'+a} = \frac{m}{r'}$ if $l = \frac{(a+r')m}{r'}$. Hence

$$(2.14) \quad \frac{a}{\varphi(a)} \left(\sum_{i=0}^{H-1} \frac{l-m}{r'+ai} + \frac{l}{r'+aH} \right) = \frac{al}{r'\varphi(a)}$$

if $l = \frac{(a+r')m}{r'}$.

Now by (2.3), (2.10), (2.13) and (2.14), we can deduce from $H = K_{r'}$ that

$$\begin{aligned} (2.15) \quad \log L_{m,l}(n) &= \sum_{r \in R(a)} \sum_{\substack{p \in P_{m,l}(n) \\ p \equiv r \pmod{a}}} \log p + O(\sqrt{n}) \\ &= \frac{an}{\varphi(a)} \left(\sum_{r \in R(a)} A_{r'} \right) + O(n \exp(-c_1 \sqrt{\log n})), \end{aligned}$$

where $A_{r'}$ is defined as in (1.3).

Since r' runs over $R(a)$ as r does, we have

$$\sum_{r \in R(a)} A_{r'} = \sum_{\substack{r=1 \\ \gcd(a,r)=1}}^a A_r.$$

It then follows immediately from (2.15) that

$$\log L_{m,l}(n) = \frac{an}{\varphi(a)} \left(\sum_{\substack{r=1 \\ \gcd(a,r)=1}}^a A_r \right) + o(n)$$

as desired. This completes the proof of Theorem 1.1. \square

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